

Global effects in quaternionic quantum field theory

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Abstract

We present some striking global consequences of a model quaternionic quantum field theory which is locally complex. We show how making the quaternionic structure a dynamical quantity naturally leads to the prediction of cosmic strings and non-baryonic hot dark matter candidates.

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I. INTRODUCTION

Of the various classes of phenomenological models and theoretical constructs extending the standard model, one which has received not much attention is quaternionic quantum mechanics.

This is curious as the theory proceeds from a firm axiomatic basis [1], and previous researchers have made some profound claims for its physical content including early models of unification of fundamental forces [2], algebraic confinement of quarks or preons [3], new effects in particle interferometry experiments [4,5], and quaternionic effects in multi-particle correlated systems [6]. Further, the theoretical construction is sufficiently restricted by general mathematical principles that model building is constrained and the theory can be predictive.

Important early work on the formulation of quaternionic quantum mechanics and field theory was due to Finkelstein, Jauch, Schiminovich and Speiser [2], who introduced the idea of a gauge sector arising from a local, quaternionic structure in space-time. At the time of this proposal (the early 60's), the motivation was to achieve a unification of electromagnetism with isospin. With the success of the Salam–Weinberg electroweak unification, interest in this work faded. When divorced of its particle-physics interpretation, however, the idea of introducing a locally complex description of nature has several interesting consequences, which we shall develop in the next section.

Our model is based on a theory which is locally equivalent to the standard, complex formalism, with the essentially quaternionic degrees of freedom appearing as an additional, exotic gauge sector. Interaction with the Standard Model sector is constrained by the fact that the new gauge fields are quaternion imaginary. Hence, minimal coupling procedures fail, but these can be coupled to normal matter via non-renormalisable terms in an effective Lagrangian. While this can be criticised for being too naïve a construction, we wish to bring our theory into the realm of experimentally constrainable physics.

On this occasion, commemorating the 65th birthday of L. P. Horwitz, we feel it appro-

priate to add a personal note. After the work of Finkelstein *et al.*, the next application of quaternionic quantum theory was proposed by Alder [3], who raised the possibility of an algebraic mechanism for confinement in hadronic physics. Attention focussed on the properties of many-body systems in a quaternionic framework. A major investigation of the theory of tensor products in quaternionic quantum mechanical systems was the classic paper of Horwitz and Biedenharn [7], which analysed the problems associated with forming tensor products of quaternion valued single-particle wavefunctions. This formalism, developed further in [8], lead the present authors to consider afresh the question of uniquely quaternionic phenomena in quantum mechanical systems, the result of which may be found in [6], on a new effect in multi-particle correlated systems. The impact of [7] and it's successors on the quaternionic community has been profound, and it is a privilege for us to contribute a paper to this volume.

II. THE MODEL

Studies of quaternionic quantum mechanics and field theory start with a Hilbert space over the quaternion algebra. The eigenvalues of operators need not commute, but physical observables continue to be identified with Hermitian operators which, as is well known, necessarily have real eigenvalues only. (It is this fact which permits the existence of quaternionic quantum mechanics.)

The relation between kets and quaternion-valued wavefunctions follows the usual pattern (where the Φ_i are real),

$$\Phi(x_\mu) \equiv \langle x_\mu | \Phi \rangle = \Phi_0 + \Phi_1 i + \Phi_2 j + \Phi_3 k. \quad (1)$$

If we define $|\vec{\Phi}| = (\sum_{r=1}^3 \Phi_r^2)^{-1/2}$, then there exists a pure imaginary quaternion of norm unity, η_Φ , such that

$$\Phi(x_\mu) = \Phi_0(x_\mu) + |\vec{\Phi}(x_\mu)| \eta_\Phi(x_\mu). \quad (2)$$

From this we see that any two wavefunctions do not commute unless their imaginary parts are parallel. This property has confounded the construction of a completely satisfactory tensor product, i.e., one which would be quaternion linear in each factor, allow the definition of a tensor product of operators on each factor, and admitting a positive scalar product for the purpose of second quantisation of the theory (see [8,9] for comprehensive discussions).

Alternatively, we can express Φ as an ordered pair of i -complex numbers, in its so-called symplectic representation,

$$\Phi(x_\mu) = \phi_\alpha + j\phi_\beta, \quad (3)$$

where $\phi_\alpha = \Phi_0 + i\Phi_1$ and $\phi_\beta = \Phi_2 - i\Phi_3$.

Such a representation implicitly breaks the full quaternionic symmetry of the theory. Given the requirement of eventually connecting with standard (complex) phenomenology, there is a strong desire to introduce such a breaking early in the theoretical formulation, often with a caveat that sufficiently early in the history of the universe the full quaternionic symmetry is to be restored. Justification for introducing such a breaking follow from arguments about the structure and symmetries of the equations of motion, from the fact that a clever definition of the order of multiplication can make the second symplectic component (i.e., the ϕ_β term) decay exponentially outside of interaction regions (making the theory asymptotically i -complex), and from consideration of linearity properties of tensor products.

We shall instead start with a model which, we maintain, retains the spirit of the original while holding out a strong hope of connecting with experimental physics. We begin by summarising the locally complex theory of Finkelstein *et al.*, but with our own interpretation, and with an important addition.

Hence, consider a space-time on which there is defined a set of scalar fields,

$$i(x_\mu), \quad j(x_\mu), \quad \text{and} \quad k(x_\mu) \equiv i(x_\mu)j(x_\mu), \quad (4)$$

which locally define a basis for the set of pure imaginary quaternions,

$$i^2(x_\mu) = -1, \quad j^2(x_\mu) = -1, \quad \text{and} \quad \{i(x_\mu), j(x_\mu)\} = 0. \quad (5)$$

We claim that the overwhelming success of conventional, complex quantum mechanics and quantum field theory is strongly indicative of the validity of a complex description of local physics (at least, on scales presently accessible to laboratory physics). Hence, in our model the matter and gauge fields will be assumed to be $i(x)$ -complex, i.e., amplitudes are real linear superpositions of 1 and the $i(x_\mu)$ field, and transform as vectors under a local quaternionic gauge symmetry,

$$\phi(x_\mu) \rightarrow q(x_\mu)\phi(x_\mu)q(x_\mu)^{-1}, \quad (6)$$

where q is a pure imaginary quaternion of unit magnitude.

The gauge theory arising from this construction is SU(2) like. We claim that this is a new sector; we do not identify it with the fundamental SU(2) in electroweak theory, nor with any approximate SU(2) in hadronic physics. Instead, we follow the tradition of identifying essentially quaternionic physics with an outstanding puzzle of contemporary physics, and claim that this new sector is a source of non-baryonic dark matter in the universe. Further, we shall show how the breaking of this local symmetry leads to cosmic strings.

The gauge-covariant derivative follows the usual pattern of Yang-Mills theory, but with a quaternionic valued potential,

$$D_\mu\phi = \partial_\mu\phi + \frac{1}{2}(Q_\mu\phi - \phi Q_\mu), \quad (7)$$

$$Q_\mu(x) \equiv \bar{A}_\mu(x)i(x) + B_\mu(x)j(x) + B'_\mu(x)k(x). \quad (8)$$

Our choice of notation follows, *a posteriori*, from our interpretation of the phenomenological implications of our model.

Finally, note that $D\phi$ reduces to $\partial\phi$ when ϕ is real.

Now we construct the field strength tensor,

$$K_{\mu\nu} = D_\mu Q_\nu - D_\nu Q_\mu, \quad (9)$$

and exhibit the Lagrangian density for the new sector, which treats i , j and k symmetrically,

$$\mathcal{L}_Q = \frac{1}{4}K^{\mu\nu}K_{\mu\nu} + \frac{\lambda}{2}|D_\mu i|^2 + \frac{\lambda'}{2}|D_\mu j|^2. \quad (10)$$

This Lagrangian differs from that of Finkelstein, *et al.* [2] by the addition of the final term. This requires several comments.

Firstly, it is important to notice that the i and j fields are dimensionless. Scale is introduced by the constants λ , λ' , which are independent in the most general case, and have the dimension of mass squared. In this regard, we might interpret the work of Finkelstein *et al.* as being in the $\lambda' \rightarrow 0$ limit.

As written, \mathcal{L}_Q is composed of dimension 4 operators. The components of the Q potential have the canonical dimension of vector bosons. We shall see that the presence of the dynamical terms $|Di|^2$ and $|Dj|^2$ inevitably give the bosons mass, and there is no symmetric phase in which to renormalise the theory. That is, the magnitudes of i and j are fixed, and our theory is always in the broken phase in which the massive vector bosons have (from the UV behaviour of bosonic propagators) anomalous dimensions which renders the theory non-renormalisable [10]. We do not regard this as catastrophic, our i and j fields have not been quantised, so our theory is of a semiclassical type. A fully quantised theory of a dynamical quaternionic structure has yet to be developed, and we concur with [2] that any such theory will share similarities with attempts to quantise gravity. In fact, interest in quantum gravity lead us to this field.

We have not included a $(\lambda''/2)|D_\mu k|^2$, as at each point of space-time a consistent definition of the quaternionic algebra requires $k(x) \equiv i(x)j(x)$, so there are no additional dynamical degrees of freedom.

In order to interpret the physical content of our model, we make a particular local quaternionic gauge transformation, to what we call the i gauge (strongly analogous to the unitary gauge of electroweak physics). Now there always exists a quaternionic gauge transformation taking $i(x_\mu)$ to a particular pure imaginary unit quaternion i , $q_i(x_\mu) i(x_\mu) q_i(x_\mu)^{-1} = i$ [2].

In this particular gauge, the quaternionic potential is

$$Q_\mu(x) = \bar{A}_\mu(x) i + (B_\mu(x) + B'_\mu(x) i) j'(x), \quad (11)$$

where $j'(x) = q_i(x_\mu) j(x) q_i(x_\mu)^{-1}$ is defined up to a real phase,

$$j'(x) = \exp[\vartheta(x) i] j = \cos \vartheta(x) j + \sin \vartheta(x) k. \quad (12)$$

That is, we have introduced a fixed basis i , j and $k = ij$, and found that the remaining degrees of freedom reside in the phase of j' , which is a real Goldstone boson, and in the transformed vector potentials \bar{A} , B and B' . The Lagrangian has become

$$\mathcal{L}_Q = \frac{1}{4} K^{\mu\nu} K_{\mu\nu} + \frac{\lambda}{2} \left| [B_\mu j' + B'_\mu i j', i] \right|^2 + \frac{\lambda'}{2} |D_\mu j|^2. \quad (13)$$

$$\begin{aligned} &= -\frac{1}{4} (\partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu + B_\mu B'_\nu - B_\nu B'_\mu)^2 + \{\text{permutations of } \bar{A}, B \text{ and } B'\} \\ &+ \frac{\lambda}{2} (B_\mu^2 + B_\mu'^2) + \frac{\lambda'}{2} (\partial_\mu \vartheta + \bar{A}_\mu)^2 + \frac{\lambda'}{2} (B_\mu \sin \vartheta - B'_\mu \cos \vartheta)^2. \end{aligned} \quad (14)$$

This particular gauge choice permits an interpretation of the physical content of the theory.

First of all, we see that the dynamical nature of the i field has given degenerate masses to the B and B' fields, in a type of spontaneous symmetry breaking noticed by [2]. If only the i field is regarded as being dynamical (the $\lambda' \rightarrow 0$ limit), then the $SU(2)$ gauge symmetry is broken to an (Abelian) \bar{A} . (A difference in the scales $\lambda > \lambda'$ will have consequences for the phenomenology of our theory, and will be briefly discussed in the next section.) Cosmological relic populations of these B and B' fields will constitute warm dark matter candidates, as discussed in [11]. Also note that the degeneracy of the masses is essentially due to the singling out of i as a special pure imaginary quaternion.

Note that symmetry breakdown has occurred via a choice of gauge, which we maintain has to happen to remove the unphysical degrees of freedom from the free field Lagrangian. So a locally complex quaternionic field theory has observational consequences which mimic those of an exotic $SU(2)$ gauge theory in the usual complex framework, except that, on general grounds, coupling of this sector to the standard sector is restricted to take a special form described in the next section, and we find global phenomena in the form of topological defects, as we now show.

The implications of the dynamical j field are brought out most clearly if we consider the Abelian restriction of our theory (the “electromagnetic” limit of [2]), which is motivated by requiring that $Di = 0$, in analogy with the consistency conditions imposed on the Levi-Civita connection in classical general relativity ($\nabla g_{\mu\nu} = 0$, see [13]). Then the B and B' fields vanish identically, and we have the free field Lagrangian

$$\mathcal{L}_{Q\bar{A}} = -\frac{1}{4}F_{\bar{A}}^{\mu\nu}F_{\bar{A}\mu\nu} + \frac{\lambda'}{2}(\partial_\mu\vartheta + \bar{A}_\mu)^2, \quad (15)$$

which has the pattern of the Abelian Higgs model [14], without the usual polynomial potential term. But our j field is defined to be of unit magnitude, and so we automatically have a vacuum expectation value for this field, which breaks the remaining \bar{A} Abelian symmetry.

The consequences of this are that the \bar{A} picks up a mass $\sqrt{\lambda'}$, and our theory supports cosmic string solutions of the classic Nielsen–Olesen type. That is, treating the j field as a dynamical quantity on the same footing as the i field completes the breaking of the local, quaternionic gauge symmetry, and introduces the possibility of global phenomena of a topological character. While cosmic strings might not be the most popular dark matter candidate at the present date, it is interesting to see them arising in this context.

We contrast our treatment with the original of Finkelstein *et al.*, in which the electromagnetic limit removed all global effects. Such a construction can be criticised as being too close to the standard complex theory, but suited well the motivations of the day.

III. INTERACTION WITH VISIBLE SECTOR

We now seek to couple this exotic sector to the locally complex fields of the Standard Model. We shall present our interaction Lagrangian, and discuss its physical content. Detailed calculations using this model are reported elsewhere [11].

We follow the general discussion of [12], which discusses possible interaction Lagrangian terms in the context of axionic physics. In our case, Q and K are pure imaginary quaternion, which leads us to propose an effective interaction Lagrangian which couples field densities

to K^2 ,

$$\mathcal{L}_{\text{int}} = \frac{g_Q}{4\Lambda_Q^2} \phi^\dagger \phi K^2 + \frac{g_Q}{4\Lambda_Q^3} \bar{\psi} \psi K^2 + \frac{g_Q}{16\Lambda_Q^4} \text{tr} F^2 K^2. \quad (16)$$

The exclusion of terms such as $\bar{\psi} \sigma \psi \cdot K$ is a constraint on model building imposed by the framework of our theory.

Here, λ and λ' are *a priori* different. and we propose that $\lambda' \ll \lambda$. Then in a low to middle energy system, we might expect \bar{A} production to dominate.

The scale Λ_Q is a characteristic quantity in QQFT, with a role that is equivalent to that of Λ_{QCD} in the $\overline{\text{MS}}$ renormalisation schemes. Our theory is only effective. Consideration of the underlying fundamental theory is of paramount interest, but is expected to give equivalent results to the effective theory below the Λ_Q scale. It is natural to expect Λ_Q to be somewhere between the electroweak and Planck scales, but we do not rule out the possibility that this effective description may be inadequate for a description of top quark physics. This would place nontrivial predictions of our theory in an experimentally interesting region. These interaction terms are contact terms, describing mutli-particle production above a mass pole (governed by λ') with Λ_Q^{-1} our perturbation parameter (more precisely, some relevant mass $m \ll \Lambda_Q$ allows us to take m/Λ_Q as our parameter).

Note that several generalisations of the Standard Model immediately suggest extensions of the above scheme. For example, similar couplings of SUSY partners to SM fields, or models with horizontal symmetry. We leave development of these ideas for the future.

Our interpretation of the physical content of this model is consistent with the introduction of non-baryonic dark matter. Coupling of normal matter to the new fields is suppressed by the Λ_Q scale, and it is possible to investigate the implications of this dark matter in astrophysical systems and at cosmological scales. Elsewhere, we present a detailed discussion of such an investigation.

IV. SUMMARY AND CONCLUSION

We have introduced a local quaternionic gauge structure onto space-time. Treating i and j as dynamical fields breaks this symmetry completely, leaving three massive vector bosons. Cosmic string solutions exist, arising from the orientation of j .

Our theory is non-renormalisable. It is a theory of vector bosons and dimensionless scalar fields, which recalls semi-classical treatments of gravity. After transforming to the i gauge, we find that the quaternionic symmetry takes the form of an exotic $SU(2)$ gauge theory in the standard complex framework, with global phenomena appearing in the form of cosmic strings.

Coupling this quaternionic sector to the Standard Model sector has only been achieved at the level of an effective theory, which is constrained by the quaternionic origin of the exotic bosons to be of a nonrenormalisable form.

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